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Multi-dimensional simple waves in fully relativistic fluids

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Abstract
Multi-dimensional simple waves built on the basis of the Boillat condition are employed for fully relativistic fluid and plasma flows. Three essential modes, namely the vortex, the entropy and sound modes, are derived where each of them is different from its non-relativistic analogue. The vortex and entropy modes are formally solved in both the laboratory and wave frame (co-moving with the wavefront) to yield a few classical solutions. It is shown that by making a suitable transformation to the wave frame, it will be possible to exactly solve the sound mode at ultra-relativistic temperatures. In addition, the surface which is the boundary between the permitted and forbidden regions of the solutions is introduced and determined. Finally, a symmetry analysis is performed for the vortex mode equation up to both point and contact transformations. The derived symmetry properties and their corresponding fundamental invariants are shown to create a wide variety of classical solutions; some of them may have physical interest.

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1. Introduction

Investigation of the nonlinear phenomena appearing in a very wide area of pure and applied sciences has undergone extensive progress and development. These studies, which split into numerical and analytical considerations, are essential and in most cases related to some nonlinear differential equations. In spite of such a huge amount of improvements, almost all of these equations are still far from being well understood. Among those nonlinear systems, a few interesting open problems concern the hydrodynamic type of equations governing the fluid motion. In particular, the Euler and Navier–Stokes equations, which exhibit a mysterious behaviour, are being intensively studied in two main considerations: the incompressible motion mostly dealing with the vortex dynamics and the compressible flow concerning the appearance of discontinuous shocks. Both of these problems are somehow related to the debate on the regularity of solutions. In the former consideration the aim is to understand the mechanism of occurrence of vortex singularities, while in the latter the shock convergence is the most important question [1–3]. The core of these subjects is the following principal open question: starting from an initially smooth flow, how can we predict the appearance of any kind of discontinuity or singularity in later times?

The wide variety of applications to fluid motions makes it necessary to deal with all kinds of differential equations, namely, elliptic, parabolic and hyperbolic equations. Elliptic and parabolic equations have relatively better behaviour than hyperbolic ones and thus regularity results of the former equations are stronger and more developed relative to the latter ones. Indeed, there are many well-defined methods for hyperbolic conservation equations, for example, well-posedness results in the form of formal mathematical proofs [4]. However, the breakdown of solutions in hyperbolic systems is a very open and complex problem. Only in some cases for dissipative hyperbolic systems do we know that there exist global smooth solutions provided initial data are sufficiently small while the necessary condition is still an open problem also in semi-linear cases (which may include hydrodynamic-type equations). These equations posses some characteristic curves or surfaces which naturally have the capability of forming discontinuities. This, of course, makes the nature of the classical (i.e. continuously differentiable or $\mathcal{C}^1$) solution to be 'local', which means that the classical solution may exist only in some part of the space and in some intervals of time. Generally, there are many topological, geometrical and analytical unknown features determining the validity of any solution which is very difficult to discover with the present human knowledge, and thus new tools are needed.

In the present paper, a classical ($\mathcal{C}^1$) solution for relativistic flows is addressed. This type of solution is not only interesting in itself, but it also seems that it may play an important role in developing perhaps a new type of weak solutions. Indeed, although generally in proofs of the existence and uniqueness of weak solutions we do not need any classical solution, sometimes it is useful to build a 1D shock convergence proof based on a 1D Riemann or simple wave (classical) solution [1]. Simple (Riemann) waves lie in the framework of an excellent and rich set of classical solutions for compressible ideal flows governed by hyperbolic equations. They have the capability of breakdown and discontinuity formation [1, 2] and blow-up occurrence [5]. Simple waves are constructed on the basis of characteristics which clearly are not global classical solutions. They were discovered first by Riemann in the 1D form [6–12] and still appear as very common and useful analytical tools to achieve shock waves [1, 2, 7, 9, 13, 14]. By imposing more restrictions and limitations on these solutions it was possible to build multi-dimensional simple waves [15–20]. Even a more generalized treatment yields double waves and multi-waves, which yield more advanced classical solutions with more intensive blow-up [20–28]. It looks acceptable that one can obtain a new type of shock convergence built on the basis of multidimensional simple waves presented here.

Although boundary conditions are usually important for any system of differential equations but since we consider here no boundary data for our problem, we have the benefit of using symmetry analysis for the equations under study. Symmetry methods have been already employed for studying fluid equations mostly in non-relativistic fluids. Hence, it is natural to ask about these methods in solving equations (6)–(9). However, this looks like a very difficult task and an easier way is to investigate (restricted) simple wave modal equations through symmetry analysis. This is done in the present study only for the vortex mode for simplicity. Symmetry investigations reach a wide variety of information about equations under consideration some of which may be physically of interest. For instance it is possible to find the Lie point and contact groups of transformations which preserve the submanifold of the classical solutions of the equation. This leads to obtain a new solution based on an existing solution (see discussions related to equation (115) for details). Another advantage of symmetry methods lies within finding invariants of group actions which finally tends to extract further solutions (see section 5.2).

It is obvious that taking into account relativistic effects highly increases the coupling and so the nonlinearity of the fluid and plasma motions. Relativistic flows have been known for a long time [29–31], and especially they are important in astrophysical and cosmological phenomena. In addition, under recent technical progresses in intense laser–plasma interactions, in the plasma-based high-energy charged particle acceleration schemes and in fusion plasmas, the access to relativistic effects in the laboratory is now very easy. Therefore, great attention has been paid to analyse relativistic flows in plasmas. Again, the study of simple waves plays a very fundamental role as almost the only available nonstationary exact solution with the ability of discontinuity formation.

A very excellent and complete mathematical discussion on one-dimensional relativistic MHD simple waves has been reviewed by Shikin [32]. Some solutions of these 1D simple waves are found in many papers. Although a relativistic 2D double wave solution has been given only for ultra-relativistic fluids [27], still we observe the missing of a multi-dimensional simple wave for a fully relativistic flow. This task is the aim of the present paper in which the approach of [19] is employed and generalized.

Physically, it is a valid question to ask why we should consider relativistic fluids while usually at such high temperatures matter is found in the plasma form and so one has to take into account electromagnetic fields leading to MHD equations. However, this is not always true because sometimes we deal with neutral fluids like neutron stars. Moreover, in the absence of any external magnetic field and when the typical length and time for the non-neutrality of the plasma are sufficiently less than the length and time for macroscopic motions, the plasma can be considered as a neutral fluid with a very high accuracy. Hence, it makes sense to consider the ideal relativistic flow here.

In the next section, after a brief derivation of relativistic ideal fluid equations, a multi-dimensional simple wave ansatz is substituted into these equations and various modes and phase velocities relative to the laboratory (fixed) frame are found. In section 3, some solutions for the vortex and entropy modes are given only in the laboratory frame. The presented solutions are very general and formal, and a detailed solution is very difficult and needs to determine the initial and boundary conditions precisely. Thus, our solutions are very general including many arbitrary functions. In section 4, the equations are rewritten in the wave frame and again some simple typical solutions are given for the all three modes, namely the vortex, the entropy and sound modes. In particular, for the sound mode, since its equations are very complicated in the laboratory frame, we will see in section 4 that these equations in the wave frame at ultra-relativistic temperatures are simplified, and it will be possible to obtain formal solutions for it. In section 5, we investigate symmetry properties and their related topics for the vortex mode equation as a sample equation appearing in our problem. Finally, a summary and concluding remarks are given in section 6.

2. Multidimensional simple wave formulation

Relativistic effects in continuum matters occur in two aspects, namely large macroscopic (fluid) velocities and relativistic temperatures at which the mean thermal energy of particles are comparable with their rest energy. Both of these aspects are included in the energy–momentum tensor

$$T_i^k = wu^k u_i - P\delta_i^k \qquad (i, k = 0, 1, 2, 3), \tag{1}$$

where $u^j = (\gamma, \gamma\mathbf{v}/c)$ is the contra-variant 4-velocity and thus $u_j = (\gamma, -\gamma\mathbf{v}/c)$ is the co-variant 4-velocity and $w = \varepsilon + P$ in which P is the fluid pressure and ε is the internal energy (including the rest energy) per unit proper volume (unit volume in the inertial frame in which the fluid is momentarily at rest). Therefore, w is the enthalpy per unit proper volume. Also, $\mathbf{v}$ is the fluid velocity and $\gamma = (1 - v^2/c^2)^{-1/2}$ where c is the speed of light in vacuum.

Our basic equations consist of the continuity equation

$$\frac{\partial}{\partial x^i}(nu^i) = 0, \qquad \text{or} \qquad \frac{1}{c}\frac{\partial}{\partial t}(\gamma n) + \nabla \cdot (n\gamma\mathbf{v}) = 0, \tag{2}$$

and the vanishing 4-divergence of the energy–momentum tensor

$$\frac{\partial}{\partial x^k}T_i^k = 0 \qquad (i = 0, 1, 2, 3) \tag{3}$$

where n is the number density of the fluid particles in the proper frame and (x^i) is the 4-vector of the spacetime coordinates in which $x^0 = ct$ (c is the speed of light) and $(x^1, x^2, x^3) = \mathbf{r}$. By virtue of the thermodynamic identity, $T\mathrm{d}S = \mathrm{d}(w/n) - \mathrm{d}P/n$ (T is the fluid temperature and S is the entropy per unit particle), one can combine equations (2) and (3) to obtain [32]

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial t} + \mathbf{v}\cdot\nabla S = 0. \tag{4}$$

Equation (4) can be alternatively considered in place of the zeroth component ($i = 0$) of equation (3). Thus, our set of equations consists of five equations (2), (4) and the spatial components of (3). This system, of course, needs a thermodynamical equation of state $P = P(S, w)$. However, it was found that this system of equations takes a more appropriate form by the use of the following useful transformation [32]:

$$\kappa^i = \frac{w}{mnc}u^i, \qquad \tilde{\rho} = \frac{(mnc)^2}{w}, \tag{5}$$

where m is the 'mean' rest mass of all particles in the fluid. This transformation makes our final system of equations to

$$\frac{1}{c}\frac{\partial}{\partial t}(\tilde{\rho}\kappa_0) + \nabla\cdot(\tilde{\rho}\boldsymbol{\kappa}) = 0, \tag{6}$$

$$\frac{\kappa_0}{c}\frac{\partial\boldsymbol{\kappa}}{\partial t} + (\boldsymbol{\kappa}\cdot\nabla)\boldsymbol{\kappa} = -\frac{1}{\tilde{\rho}}\nabla P, \tag{7}$$

$$\frac{\kappa_0}{c}\frac{\partial S}{\partial t} + \boldsymbol{\kappa}\cdot\nabla S = 0, \tag{8}$$

$$P = P(S, \tilde{\rho}). \tag{9}$$

Here $\kappa^i = (\kappa^0, \boldsymbol{\kappa})$, and thus $\kappa_i = (\kappa_0, -\boldsymbol{\kappa})$ and

$$\kappa^0 = \kappa_0 = \sqrt{\kappa^2 + w/\tilde{\rho}} \qquad (\kappa = |\boldsymbol{\kappa}|), \tag{10}$$

which follows from the identity $u^i u_i = 1$.

The special feature of a simple wave in any quasilinear hyperbolic system of equations is that all quantities are considered as functions of only one variable which we call the phase and denote by $\varphi = \varphi(\mathbf{r}, t)$. In our problem, we write

$$\mathbf{U} = \mathbf{U}(\varphi), \tag{11}$$

where

$$\mathbf{U} = (\tilde{\rho}, \kappa_1, \kappa_2, \kappa_3, S) \tag{12}$$

is the state vector of the system. Boillat [17, 19] showed that for a simple wave it is necessary to have

$$\frac{\nabla\varphi}{|\nabla\varphi|} := \mathbf{n} = \mathbf{n}(\varphi), \qquad -\frac{\partial\varphi/\partial t}{|\nabla\varphi|} := \lambda = \lambda(\varphi). \tag{13}$$

In other words, the unit vector $\mathbf{n}$ normal to the wavefront must be only a function of φ and the same is true for the phase velocity λ. Finally, condition (13) implies that φ must satisfy [17, 19]

$$G(\varphi, \mathbf{r}, t) := f(\varphi) + \lambda(\varphi)t - \mathbf{r}\cdot\mathbf{n}(\varphi) = 0, \tag{14}$$

where f is an arbitrary differentiable function to be fixed under initial conditions. Equation (14) clearly means that level surfaces of φ are flat planes. The functional form of $\mathbf{n}(\varphi)$ cannot be determined from any equation and so it remains arbitrary to be flexible to fit under a given condition.

There are two noticeable points about our multi-dimensional simple waves. The first is the wave breaking at which the time and spatial derivatives of φ and all variables diverge when $F \longrightarrow 0$ provided that

$$\frac{\partial\varphi}{\partial t} = -\frac{\lambda(\varphi)}{F}, \qquad \nabla\varphi = \frac{\mathbf{n}(\varphi)}{F}, \tag{15}$$

$$F := \frac{\partial G}{\partial\varphi} = \frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} + \frac{\mathrm{d}\lambda(\varphi)}{\mathrm{d}\varphi}t - \mathbf{r}\cdot\frac{\mathrm{d}\mathbf{n}(\varphi)}{\mathrm{d}\varphi} = \frac{1}{|\nabla\varphi|}. \tag{16}$$

Equations (15) are easily derived by implicit time and space differentiations of (14). Thus, our simple wave solution is valid only when $F > 0$ and at any time and point where $F = 0$ solution is not correct. The second point arises from the dependence of $\mathbf{n}$ on φ which implies that for two different values φ_1 and φ_2 of φ generally $\mathbf{n}(\varphi_1)$ and $\mathbf{n}(\varphi_2)$ are not parallel and thus they have an intersection on a line at which the solution is multi-valued which is not accepted. Hence, the domain of the valid solution must not contain such intersections. Both of these points demonstrate the 'local' character of simple waves.

For a unidirectional 1D simple wave where $\mathbf{n}$ is a constant vector it is possible for each value of φ to calculate the time of wave breaking ($F = 0$) as $t_c(\varphi) = -(\mathrm{d}f/\mathrm{d}\varphi)/\mathrm{d}\lambda/\mathrm{d}\varphi$ and the earliest time of the wave breaking is obtained by solving the equation $(\mathrm{d}t_c/\mathrm{d}\varphi) = 0$ [9]. Unfortunately, such a good situation does not hold in the multidimensional case when $\mathbf{n} = \mathbf{n}(\varphi)$. Let us see this in a quantitative way. Singular points (wave breaking) must not only satisfy the simple wave condition (14), but also they should fulfil

$$F = 0. \tag{17}$$

Thus, the wave breaking occurs on the line of intersection of the two perpendicular planes $G = 0$ and $F = 0$. This line is exactly the rotation axis of the wavefront at φ when φ has an infinitesimal growth to $\varphi + \delta\varphi$. This will be easily seen if we observe that the wavefront at $\varphi + \delta\varphi$ must satisfy

$$G(\varphi + \delta\varphi, \mathbf{r}, t) = 0, \qquad \text{or} \qquad G(\varphi, \mathbf{r}, t) + F\delta\varphi = 0.$$

Since the above equation holds for any value of $\delta\varphi$, equations (14) and (17) appear again. We may, therefore, conclude (without a rigorous proof) that if the wave breaking (singularity) line lies out of the region of the solution, the line of multi-valuedness will also lie in that region. Besides, since a line of singularity for each value of φ exists at each instant of time, it is infeasible to speak about $t_c(\varphi)$. However, if the fluid fills the whole space $\mathbb{R}^3$ we can obtain a moving surface constructed at any time exactly from all of these singular lines at that time. This surface is in fact the boundary between the forbidden and permitted regions relative to a simple wave solution.

Now, we substitute the simple wave ansatz (11) into equations (6)–(8) supplemented by equation (9) and then divide each equation by $|\nabla\varphi|$ and use (13) to obtain the system of five quasilinear coupled equations

$$A\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\varphi} = 0, \tag{18}$$

where A is the 5×5 matrix with the following elements:

$$A = \begin{bmatrix} \kappa_n - \frac{\lambda}{c}\left(\tilde\rho\frac{\partial\kappa_0}{\partial\tilde\rho}+\kappa_0\right) & \tilde\rho\left(n_1 - \frac{\lambda}{c}\frac{\kappa_1}{\kappa_0}\right) & \tilde\rho\left(n_2 - \frac{\lambda}{c}\frac{\kappa_2}{\kappa_0}\right) & \tilde\rho\left(n_3 - \frac{\lambda}{c}\frac{\kappa_3}{\kappa_0}\right) & -\frac{\lambda}{c}\tilde\rho\frac{\partial\kappa_0}{\partial S} \\ \frac{a^2 n_1}{\tilde\rho} & \kappa_n - \frac{\lambda}{c}\kappa_0 & 0 & 0 & \frac{P_S n_1}{\tilde\rho} \\ \frac{a^2 n_2}{\tilde\rho} & 0 & \kappa_n - \frac{\lambda}{c}\kappa_0 & 0 & \frac{P_S n_2}{\tilde\rho} \\ \frac{a^2 n_3}{\tilde\rho} & 0 & 0 & \kappa_n - \frac{\lambda}{c}\kappa_0 & \frac{P_S n_2}{\tilde\rho} \\ 0 & 0 & 0 & 0 & \kappa_n - \frac{\lambda}{c}\kappa_0 \end{bmatrix}. \tag{19}$$

In the above matrix, we have used the following notations:

$$\kappa_n := \boldsymbol{\kappa}\cdot\mathbf{n} = \sum_{i=1}^{3}\kappa_i n_i, \qquad a^2 := \left(\frac{\partial P}{\partial\tilde\rho}\right)_S, \qquad P_S := \left(\frac{\partial P}{\partial S}\right)_{\tilde\rho}. \tag{20}$$

Moreover, in the calculation of $\frac{\partial\kappa_0}{\partial\tilde\rho}$ and $\frac{\partial\kappa_0}{\partial S}$ we must assume $w = w(\tilde\rho, S)$ and use equation (10) to express κ_0 explicitly as a function of all five variables $\mathbf{U} = (\tilde\rho, \boldsymbol{\kappa}, S)$.

Equation (18) has a nontrivial solution only when

$$\det(A) = 0, \tag{21}$$

which constructs a fifth-order equation for λ with a triple root

$$\lambda_1 = \lambda_2 = \lambda_3 = c\frac{\kappa_n}{\kappa_0} = v_n, \tag{22}$$

while the fourth and fifth roots λ_4 and λ_5 are the larger and smaller roots of the following quadratic equation, respectively:

$$\left[\kappa_n - \frac{\lambda}{c}\left(\tilde\rho\frac{\partial\kappa_0}{\partial\tilde\rho}+\kappa_0\right)\right]\left(\kappa_n - \frac{\lambda}{c}\kappa_0\right) = a^2\left(1-\frac{\lambda}{c}\frac{\kappa_n}{\kappa_0}\right). \tag{23}$$

The triplet root is the phase velocity for the two vortex modes and one entropy, to be discussed in the next section. The roots λ_4 and λ_5 are the phase velocities for the forward and backward sound modes, respectively. Although these modes have non-relativistic analogues, they differ significantly from those calculated in the relativistic case.

Substitution of each value of the phase velocity into (18) yields some ordinary differential equations for $\mathbf{U}(\varphi)$ to be solved. For the entropy and vortex modes, these equations are not difficult and some formal solutions both in the laboratory and wave frames will be presented in sections 3 and 4, respectively. Since the equations for the sound waves are complicated in the laboratory frame, we transform to the wave frame and for a slight simplification we consider the physically common case of ultra-relativistic temperatures and present some solutions in section 4.

3. Vortex and entropy modes

If we substitute the triplet root $\lambda = c\frac{\kappa_n}{\kappa_0}$ into (18), we obtain

$$-\frac{\kappa_n}{\kappa_0}\frac{\partial \kappa_0}{\partial \tilde{\rho}}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi} + \left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)\cdot\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} - \frac{\kappa_n}{\kappa_0}\frac{\partial \kappa_0}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}\varphi} = 0, \tag{24}$$

$$a^2\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi} + P_S\frac{\mathrm{d}S}{\mathrm{d}\varphi} = \frac{\mathrm{d}P}{\mathrm{d}\varphi} = 0 \Longrightarrow P(\varphi) = \text{const}, \tag{25}$$

$$0\cdot\frac{\mathrm{d}S}{\mathrm{d}\varphi} = 0. \tag{26}$$

In equation (25) we have used the second and third equations of (20) together with (9). Equation (26) admits the two cases of constant entropy (the vortex mode) and variable entropy (the entropy mode).

3.1. Vortex modes

We have $\mathrm{d}S = 0$ or

$$S(\varphi) = \text{const}, \tag{27}$$

which, together with (25) and (9), yields the constancy of $\tilde{\rho}$, and so all thermodynamical variables. Thus, only the fluid velocity considered in $\boldsymbol{\kappa}$ and $\mathbf{n}$ change with φ where $\mathbf{n}(\varphi)$ is an arbitrary suitable function. Regarding the above results in equations (24) and (10), one can obtain the equation for $\boldsymbol{\kappa}(\varphi)$:

$$\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)\cdot\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = 0, \tag{28}$$

which must be supplemented by

$$\kappa_0 = \sqrt{\kappa^2 + w_0/\tilde{\rho}_0}, \tag{29}$$

where w_0 and $\tilde{\rho}_0$ are constant throughout the wave. The factor $\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right)$ in (28) cannot be zero because if it is zero we can take its inner product with $\mathbf{n}$ and obtain $\kappa_n^2 = \kappa^2 = \kappa_0^2$ which is impossible by (29). Therefore, equation (28) is equivalent to

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = \mathbf{X}(\varphi)\times\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right), \tag{30}$$

where $\mathbf{X}(\varphi)$ is an arbitrary continuous function. It is possible to choose two functions $\mathbf{X}_1(\varphi)$ and $\mathbf{X}_2(\varphi)$, where $\mathbf{X}_1(\varphi)\cdot\mathbf{X}_2(\varphi) = 0$, which give two perpendicular and independent vortex modes similar to the non-relativistic case [19].

It is also worth noting that we can define a generalized vortex

$$\boldsymbol{\Omega} := \nabla\times\boldsymbol{\kappa} = \nabla\varphi\times\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = |\nabla\varphi|\mathbf{n}\times\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi}, \tag{31}$$

which is constant not only on the wavefront but also in advection with the fluid velocity

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\Omega} = |\nabla\varphi|\left(\frac{c\kappa_n}{\kappa_0} - \lambda\right)\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}\varphi} = 0, \tag{32}$$

This fact is consistent with the 'frozen in' condition of field lines of $\boldsymbol{\Omega}$ [33]:

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\Omega}}{\gamma n}\right) + (\mathbf{v}\cdot\nabla)\left(\frac{\boldsymbol{\Omega}}{\gamma n}\right) = \left(\frac{\boldsymbol{\Omega}}{\gamma n}\cdot\nabla\right)\mathbf{v}, \tag{33}$$

where the number density in the laboratory frame γn is constant because

$$\nabla \cdot \mathbf{v} = c\nabla \cdot \left(\frac{\boldsymbol{\kappa}}{\kappa_0}\right) = c\frac{|\nabla\varphi|}{\kappa_0}\left(\mathbf{n} - \frac{\kappa_n}{\kappa_0^2}\boldsymbol{\kappa}\right) \cdot \frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = 0, \tag{34}$$

according to equation (28). Finally by (31) we find that

$$\boldsymbol{\Omega} \cdot \nabla = |\nabla\varphi|\boldsymbol{\Omega} \cdot \mathbf{n}\frac{\mathrm{d}}{\mathrm{d}\varphi} = 0,$$

by which equation (33) reduces to (32).

Equation (30) has many solutions since $\mathbf{X}(\varphi)$ is an arbitrary continuous function. It is, therefore, easy to choose some suitable simple forms for $\mathbf{X}$ such that equation (30) can be easily solved. As an example, consider two different forms

$$\mathbf{X} = \alpha\boldsymbol{\kappa}, \qquad \text{or} \qquad \mathbf{X} = \alpha\frac{\kappa_0^2}{\kappa_n}\mathbf{n}, \tag{35}$$

where α is a dimensionless constant. Selecting each form for $\mathbf{X}$ from (35) causes equation (30) to reduce to

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = \alpha\boldsymbol{\kappa} \times \mathbf{n}(\varphi). \tag{36}$$

Regardless of the functional form of $\mathbf{n}(\varphi)$, it is obvious from (36) that $|\boldsymbol{\kappa}(\varphi)|$ is constant and from (29) κ_0 is also constant and only the direction of $\boldsymbol{\kappa}(\varphi)$ changes by φ. To obtain a more special solution, let us consider a 2D simple wave with [19]

$$\mathbf{n}(\varphi) = (-\sin\varphi, \cos\varphi, 0), \tag{37}$$

in a Cartesian coordinate system. In this case, we have

$$\boldsymbol{\kappa} = \kappa_n\mathbf{n} + \kappa_t\mathbf{t} + \kappa_3\mathbf{z}, \tag{38}$$

where

$$\kappa_n = \boldsymbol{\kappa} \cdot \mathbf{n} = -\kappa_1 \sin\varphi + \kappa_2 \cos\varphi, \qquad \kappa_t = \boldsymbol{\kappa} \cdot \mathbf{t} = -(\kappa_1 \cos\varphi + \kappa_2 \sin\varphi), \tag{39}$$

where $\mathbf{t} = (-\cos\varphi, -\sin\varphi, 0)$ is normal to $\mathbf{n}$. Substitution of (38) into (36) and using the relations

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = \mathbf{t}, \qquad \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\varphi} = -\mathbf{n}, \qquad \mathbf{n} \times \mathbf{t} = \mathbf{z}, \tag{40}$$

one finds

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}\varphi} = \kappa_t, \tag{41}$$

$$\frac{\mathrm{d}\kappa_t}{\mathrm{d}\varphi} + \kappa_n - \alpha\kappa_3 = 0, \tag{42}$$

$$\frac{\mathrm{d}\kappa_3}{\mathrm{d}\varphi} = -\alpha\kappa_t. \tag{43}$$

Equations (41) and (43) yield

$$\kappa_3 = -\alpha\kappa_n + \bar{\kappa}, \tag{44}$$

where $\bar{\kappa}$ is a constant with the dimension of κ (velocity). Then, we substitute κ_t from (41) and κ_3 from (44) into (42) to obtain

$$\frac{\mathrm{d}^2\kappa_n}{\mathrm{d}\varphi^2} + (1 + \alpha^2)\kappa_n - \alpha\bar{\kappa} = 0, \tag{45}$$

with a general solution

$$\kappa_n = \bar{\kappa}_n \cos[\sqrt{1+\alpha^2}(\varphi+\beta)] + \frac{\alpha}{1+\alpha^2}\bar{\kappa}, \quad (46)$$

where $\bar{\kappa}_n$ is a constant with the dimension of κ (velocity), while β is a dimensionless constant. Then, from (41) we have

$$\kappa_t = -\sqrt{1+\alpha^2}\bar{\kappa}_n \sin[\sqrt{1+\alpha^2}(\varphi+\beta)], \quad (47)$$

and from (44) we find

$$\kappa_3 = -\alpha\bar{\kappa}_n \cos[\sqrt{1+\alpha^2}(\varphi+\beta)] + \frac{1}{1+\alpha^2}\bar{\kappa}. \quad (48)$$

It is then straightforward to find κ_1 and κ_2 by the use of

$$\kappa_1 = -(\kappa_n \sin\varphi + \kappa_t \cos\varphi), \qquad \kappa_2 = \kappa_n \cos\varphi - \kappa_t \sin\varphi. \quad (49)$$

It is also possible to obtain further solutions using symmetry analysis. These solutions are listed through cases I–V in section 5.2 accompanied by brief physical explanations. Another interesting fact deduced from the symmetry discussion is that if $\boldsymbol{\kappa}(\varphi)$ is a solution corresponding to a suitable form of $\mathbf{n}(\varphi)$, then one can introduce a new form for $\mathbf{n}$, say $\tilde{\mathbf{n}}$, corresponding to a new solution, say $\tilde{\boldsymbol{\kappa}}$ (see section 5.1).

As mentioned before, a complete solution needs more detailed information about initial and boundary conditions which are not of interest here.

3.2. Entropy modes

Here we have $\mathrm{d}S \neq 0$ in (26) and thus $\tilde{\rho}$ is not constant although by (25) P is still constant. According to equation (10) κ_0 is an explicit function of $\boldsymbol{\kappa}$ and $w/\tilde{\rho}$ where by the thermodynamical state equation w is a function of $\tilde{\rho}$ and S. Thus, the first and third terms on the left-hand side of equation (24) provide the derivative of κ_0 arising from the term $w/\tilde{\rho}$ and the second term in the middle bracket in equation (24) makes the derivative with respect to $\boldsymbol{\kappa}$. Therefore, equation (24) reduces to

$$\mathbf{n}\cdot\left(\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} - \frac{\mathrm{d}(\ln\kappa_0)}{\mathrm{d}\varphi}\boldsymbol{\kappa}\right) = 0, \quad (50)$$

which gives

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = \mathbf{Y}(\varphi)\times\mathbf{n} + \frac{\mathrm{d}(\ln\kappa_0)}{\mathrm{d}\varphi}\boldsymbol{\kappa}, \quad (51)$$

where $\mathbf{Y}(\varphi)$ is again an arbitrary continuous function. Let us again choose a suitable form for $\mathbf{Y}(\varphi)$ to simplify the solution. For example, if $\mathbf{Y}$ is parallel to $\mathbf{n}$ we see from (51) that

$$\boldsymbol{\kappa} = \kappa_0\mathbf{C}, \quad (52)$$

where $\mathbf{C} = (c_1, c_2, c_3)$ is a dimensionless constant vector. Since P is constant, the enthalpy w becomes only a function of $\tilde{\rho}$ and thus equations (52) and (10) yield

$$\boldsymbol{\kappa} = \frac{\mathbf{C}}{\sqrt{1-|\mathbf{C}|^2}}\sqrt{\frac{w(\tilde{\rho})}{\tilde{\rho}}}, \quad (53)$$

which is valid if $|\mathbf{C}| < 1$. It remains to specify the dependence of $\tilde{\rho}$ on φ. This dependence is arbitrary because the fixing of φ is under our control and we can assume it as an arbitrary function of one or more physical variables [19].

4. Simple waves presented in the wave frame

Sometimes the forms taken by the equations concerning simple wave solutions are quite complicated, which is to say not easily solvable. A mathematical trick here is to rewrite all equations in terms of physical variables as measured in the wave frame. The wave frame depends on a special value of the phase φ. That is, for any values of φ there is a plane wavefront defined by equation (14) moving with the phase velocity $\mathbf{V}_{\text{ph}} = \lambda(\varphi)\mathbf{n}(\varphi)$ and we consider a Lorentz transformation from the laboratory frame to the frame co-moving with this wavefront. It is thus clear that to each value of φ corresponds a unique wave frame.

We denote all quantities in the wave frame by a prime, except scalar quantities such as $\tilde{\rho}$, w, n, etc which are either Lorentz invariant or defined in the proper frame co-moving with the fluid. Therefore, for the 4-vector κ^i we have

$$\kappa_0 = \cosh\xi\kappa_0' + \sinh\xi\kappa_n', \qquad \kappa_n = \cosh\xi\kappa_n' + \sinh\xi\kappa_0', \qquad \boldsymbol{\kappa}_\perp = \boldsymbol{\kappa}_\perp', \tag{54}$$

where $\kappa_n' = \boldsymbol{\kappa}' \cdot \mathbf{n}$, $\boldsymbol{\kappa}_\perp' = \boldsymbol{\kappa}' - \kappa_n'\mathbf{n}$ and ξ depends on φ thorough

$$\tanh\xi = \frac{\lambda(\varphi)}{c}. \tag{55}$$

In the following subsections, we apply this method for the vortex, the entropy and sound modes and give simple formal solutions for each one.

4.1. Vortex modes in the wave frame

For this mode, we already have $\frac{\lambda}{c} = \frac{\kappa_n}{\kappa_0}$ which by the substitution from (54) and (55) gives

$$\kappa_n' = 0. \tag{56}$$

It is, therefore, seen that in the wave frame the phase velocity takes the simple form through the above equation. Then, we substitute (54) and (55) into (28) and use (56) and the identity $\boldsymbol{\kappa}_\perp' \cdot \mathbf{n} = 0$ to obtain

$$\cosh^2\xi\mathbf{n} \cdot \frac{\mathrm{d}\boldsymbol{\kappa}_\perp'}{\mathrm{d}\varphi} - \frac{\sinh\xi}{\kappa_0'}\boldsymbol{\kappa}_\perp' \cdot \frac{\mathrm{d}\boldsymbol{\kappa}_\perp'}{\mathrm{d}\varphi} + \sinh\xi\frac{\mathrm{d}\kappa_0'}{\mathrm{d}\varphi} + \kappa_0'\cosh\xi\frac{\mathrm{d}\xi}{\mathrm{d}\varphi} = 0. \tag{57}$$

Since $(\kappa_0, \boldsymbol{\kappa})$ is a 4-vector, equation (29) is invariant and due to (56) we have

$$\kappa_0' = \sqrt{\kappa_\perp'^2 + w_0/\tilde{\rho}_0}, \tag{58}$$

by which equation (57) reduces to

$$\boldsymbol{\kappa}_\perp' \cdot \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = -\mathbf{n} \cdot \frac{\mathrm{d}\boldsymbol{\kappa}_\perp'}{\mathrm{d}\varphi} = \frac{\kappa_0'}{\cosh\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\varphi}. \tag{59}$$

Assuming that $\mathbf{n}(\varphi)$ is a known function, $\boldsymbol{\kappa}_\perp'$ and ξ should satisfy equation (59), which obviously admits a large amount of freedom. As a very simple solution, let us assume the restriction

$$\frac{\mathrm{d}\boldsymbol{\kappa}_\perp'}{\mathrm{d}\varphi} = -\kappa_\perp'\mathbf{n}, \tag{60}$$

which dictates that $\kappa_\perp'$ and κ_0' are both constant, and thus equation (59) can be easily solved to give

$$\int_{\xi_0}^{\xi} \frac{\mathrm{d}\xi'}{\cosh\xi'} = \arctan(\sinh\xi) - \arctan(\sinh\xi_0) = \frac{\kappa_\perp'}{\kappa_0'}(\varphi - \varphi_0). \tag{61}$$

From the above solution one can find ξ in terms of φ by which through equation (55) we have $\lambda(\varphi)$. For $\boldsymbol{\kappa}'_\perp$ we formally solve equation (60) obtaining

$$\boldsymbol{\kappa}'_\perp = -\kappa'_\perp \int_{\varphi_0}^{\varphi} \mathbf{n}(\varphi')\mathrm{d}\varphi' + \boldsymbol{\kappa}'_\perp(\varphi_0). \tag{62}$$

Here we gave a very restricted solution just as an example to show the procedure of the solution. Depending on the initial and boundary conditions, it is in principle possible to find more realistic solutions although it seems to be very difficult.

4.2. *Entropy mode in the wave frame*

Since the phase velocity $\frac{\lambda}{c} = \frac{\kappa_n}{\kappa_0}$ is the same as for the vortex mode, equation (56) is again valid here and transforming equation (50) in a manner similar to that performed for the vortex mode we again obtain equation (59) for the entropy wave too but here since P is constant, w is only a function of $\tilde{\rho}$ and thus

$$\kappa'_0 = \sqrt{\kappa'^2_\perp + w(\tilde{\rho})/\tilde{\rho}}. \tag{63}$$

Hence, the meaning of (59) for the entropy mode is different from this equation for the vortex mode. Again, as a very restricted simple solution, we suggest equations (60) and (62) for $\boldsymbol{\kappa}'_\perp(\varphi)$ resulting in the constancy of $\kappa'_\perp$ and assume a given form for $\tilde{\rho}(\varphi)$ by which from (63) we have $\kappa'_0(\varphi)$ as a known function of φ and thus equation (59) has the formal solution

$$\arctan(\sinh\xi) - \arctan(\sinh\xi_0) = \kappa'_\perp \int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\varphi'}{\kappa'_0(\varphi')}. \tag{64}$$

4.3. *Sound mode in the wave frame*

At first we see that in the laboratory reference frame there are five equations included in (18) but due to equation (21) we have only four independent equations. The last (fifth) equation of (18) by the use of (19) implies that equation (27) is also valid for the sound mode which by its substitution into the first four equations of (18) yields

$$\left(\kappa_n - \frac{\lambda}{c}\kappa_0\right)\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi} - \frac{\lambda}{c}\tilde{\rho}\frac{\mathrm{d}\kappa_0}{\mathrm{d}\varphi} + \tilde{\rho}\mathbf{n}\cdot\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} = 0, \tag{65}$$

as the continuity equation, and

$$\left(\kappa_n - \frac{\lambda}{c}\kappa_0\right)\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\varphi} + \frac{a^2}{\tilde{\rho}}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi}\mathbf{n} = 0, \tag{66}$$

as the momentum equation in which a^2 is defined from (20). Equations (65) and (66) are four equations but only three of them are independent, while the phase velocity is $\lambda = \lambda_4$ or $\lambda = \lambda_5$ which are the roots of the quadratic equation (23).

Let us rewrite equations (65) and (66) in terms of the wave frame quantities through equations (54) to find

$$\kappa'_n\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi} + \tilde{\rho}\kappa'_0\frac{\mathrm{d}\xi}{\mathrm{d}\varphi} + \tilde{\rho}\frac{\mathrm{d}\kappa'_n}{\mathrm{d}\varphi} + \cosh\xi\,\tilde{\rho}\mathbf{n}\cdot\frac{\mathrm{d}\boldsymbol{\kappa}'_\perp}{\mathrm{d}\varphi} = 0, \tag{67}$$

and

$$\cosh\xi\frac{a^2}{\tilde{\rho}\kappa'_n}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi}\mathbf{n} + \frac{\mathrm{d}\boldsymbol{\kappa}'_\perp}{\mathrm{d}\varphi} + \mathbf{n}\frac{\mathrm{d}}{\mathrm{d}\varphi}(\kappa'_n\cosh\xi + \kappa'_0\sinh\xi) + (\kappa'_n\cosh\xi + \kappa'_0\sinh\xi)\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = 0, \tag{68}$$

with only three independent equations.

It is also necessary to rewrite the quadratic equation (23) (whose roots are the sound waves λ_4 and λ_5) in terms of the wave frame quantities. Substitution of (54) and (55) into (23) yields

$$\left(\kappa'_n - \tilde{\rho}\frac{\partial\kappa_0}{\partial\tilde{\rho}}\sinh\xi\right)\kappa'_n = a^2\frac{\kappa'_0}{\kappa'_0 + \kappa'_n\tanh\xi}. \tag{69}$$

Now, we note that according to equation (10) we have

$$\frac{\partial\kappa_0}{\partial\tilde{\rho}} = \frac{1}{2\kappa_0}\frac{\partial}{\partial\tilde{\rho}}\left(\frac{w}{\tilde{\rho}}\right)_S = \frac{1}{2}\frac{1}{\kappa'_0\cosh\xi + \kappa'_n\sinh\xi}\frac{\mathrm{d}}{\mathrm{d}\tilde{\rho}}\left(\frac{w}{\tilde{\rho}}\right), \tag{70}$$

where we have used equation (27) by which the entropy is constant and thus w is only a function of $\tilde{\rho}$. We then substitute equation (70) into (69) to obtain

$$\left(\kappa_n'^2 - a^2\right)\kappa'_0 + \left[\kappa_n'^2 - \frac{\tilde{\rho}}{2}\frac{\partial}{\partial\tilde{\rho}}\left(\frac{w}{\tilde{\rho}}\right)_S\right]\kappa'_n\tanh\xi = 0, \tag{71}$$

where

$$\kappa'_0 = \sqrt{\kappa'^2 + w/\tilde{\rho}}. \tag{72}$$

Equation (71) determines the phase velocity $\tanh\xi$ in terms of the physical variables measured in the wave frame but since it is generally complicated depending on the explicit form of $w(\tilde{\rho})$, we cannot go further. However, it is possible to continue for the ultra-relativistic case when $K_BT \gg mc^2$ which implies that [34]

$$\frac{w}{w_\circ} = \left(\frac{T}{T_\circ}\right)^4, \qquad \frac{n}{n_\circ} = \left(\frac{T}{T_\circ}\right)^3, \qquad P = \frac{1}{4}w,$$

by which it is easy to see

$$\frac{w}{\tilde{\rho}} = \frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}, \qquad a^2 = \frac{\mathrm{d}P}{\mathrm{d}\tilde{\rho}} = \frac{1}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}, \tag{73}$$

where the subscript '$\circ$' denotes the equilibrium point of the fluid at which it is at rest. By the above simplifications equation (71) reduces to

$$\left(\kappa_n'^2 - \frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}\right)(\kappa'_0 + \kappa'_n\tanh\xi) = 0.$$

Since $|\kappa'_0/\kappa'_n| > 1$ while $|\tanh\xi| < 1$, the second factor in the above equation cannot be zero and thus for the sound mode in the ultra-relativistic case we obtain

$$\kappa'_n = \mp a = \mp\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}}, \qquad \kappa'_0 = \sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}}. \tag{74}$$

Here the upper (minus) sign indicates the case where the fluid velocity is negative with respect to the wavefront which means that the wave runs faster than the fluid and thus it refers to the forward sound wave. Similarly, the lower (plus) sign refers to the backward sound wave.

As mentioned before, there are only three independent equations, namely equation (68) when equation (74) is substituted into it. It is more convenient to write equation (68) in the three orthogonal directions $\mathbf{n}$, $\boldsymbol{\kappa}'_\perp$ and $\mathrm{d}\mathbf{n}/\mathrm{d}\varphi$. Thus, making the scalar product of (68) (after the substitution of (74) into it) by $\mathbf{n}$ yields

$$\left(\sinh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}} \mp \cosh\xi\sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}}\right)\left(\frac{3}{2}\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}}\frac{1}{\sqrt{\tilde{\rho}\left(\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}\right)}}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\varphi} \mp \frac{\mathrm{d}\xi}{\mathrm{d}\varphi}\right)$$
$$= -\frac{\sinh\xi}{\sqrt{\kappa_\perp'^2 + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}}}\boldsymbol{\kappa}'_\perp\cdot\frac{\mathrm{d}\boldsymbol{\kappa}'_\perp}{\mathrm{d}\varphi} + \boldsymbol{\kappa}'_\perp\cdot\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi}, \tag{75}$$

where we have used the identity $\boldsymbol{\kappa'}_\perp \cdot \mathbf{n} = 0$, which gives

$$\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} \cdot \mathbf{n} + \boldsymbol{\kappa'}_\perp \cdot \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = 0. \tag{76}$$

Next, let us make the scalar product of (68) by $\boldsymbol{\kappa'}_\perp$:

$$\boldsymbol{\kappa'}_\perp \cdot \frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} = -\left(\sinh\xi\sqrt{\kappa'^2_\perp + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}^2_\circ}\tilde{\rho}} \mp \cosh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}^2_\circ}\tilde{\rho}}\right)\boldsymbol{\kappa'}_\perp \cdot \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi}. \tag{77}$$

Finally, the scalar product of (68) by $\mathrm{d}\mathbf{n}/\mathrm{d}\varphi$ is

$$\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = -\left(\sinh\xi\sqrt{\kappa'^2_\perp + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}^2_\circ}\tilde{\rho}} \mp \cosh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}^2_\circ}\tilde{\rho}}\right)|\mathrm{d}\mathbf{n}/\mathrm{d}\varphi|^2. \tag{78}$$

Thus, we should solve the system of equations (75), (77) and (78) provided that $\mathbf{n}(\varphi)$ is a known function.

Equations (77) and (78) have a common bracket on their right-hand sides which can be eliminated among these two equations. Then by using the well-known 'BAC-CAB rule' in vector triple products, one can obtain

$$\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} \cdot \left[\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} \times \left(\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} \times \boldsymbol{\kappa'}_\perp\right)\right] = 0, \tag{79}$$

which gives

$$\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} = \mathbf{Z}(\varphi) \times \left[\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} \times \left(\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} \times \boldsymbol{\kappa'}_\perp\right)\right], \tag{80}$$

where $\mathbf{Z}(\varphi)$ is an arbitrary continuous function. We will not proceed further in this way but alternatively seek more simple solutions. If we assume $\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} = 0$, then it is possible to show after some calculations that this is not a consistent solution for the system of equations (75), (77) and (78). However, a consistent simple solution is found under the assumption

$$\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} \cdot \mathbf{n} = -\boldsymbol{\kappa'}_\perp \cdot \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = 0. \tag{81}$$

This condition with the help of (77) gives

$$\kappa'_\perp = \text{const} \equiv \bar{\kappa}'_\perp. \tag{82}$$

By the use of (81) and (82), since $|\tanh\xi| < 1$, we find a differential equation relating $\tilde{\rho}$ to ξ whose solution is

$$\tilde{\rho} = \frac{\bar{\kappa}'^2_\perp\tilde{\rho}^2_\circ}{3w_\circ}\left\{\cosh\left[\pm\frac{2}{\sqrt{3}}(\xi - \xi_\circ) + \cosh^{-1}\left(1 + \frac{3w_\circ}{\bar{\kappa}'^2_\perp\tilde{\rho}_\circ}\right)\right] - 1\right\}, \tag{83}$$

where the upper (positive) sign refers to the forward and the lower (negative) sign denotes the backward sound wave. Now, we should find $\boldsymbol{\kappa'}_\perp$. It is clear from (81) and (82) that $\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi}$ is perpendicular to both $\boldsymbol{\kappa'}_\perp$ and $\mathbf{n}$; thus $\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi}$ is parallel to $\mathbf{n} \times \boldsymbol{\kappa'}_\perp$. On the other hand, the identity $\mathbf{n} \cdot \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi} = 0$ and equation (81) imply that $\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi}$ is also parallel to $\mathbf{n} \times \boldsymbol{\kappa'}_\perp$. Therefore, we conclude that

$$\frac{\mathrm{d}\boldsymbol{\kappa'}_\perp}{\mathrm{d}\varphi} = \pi(\varphi)\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\varphi},$$

or equivalently

$$\boldsymbol{\kappa'}_\perp(\varphi) = \int_{\varphi_\circ}^{\varphi} \pi(\varphi')\frac{\mathrm{d}\mathbf{n}(\varphi')}{\mathrm{d}\varphi'}\mathrm{d}\varphi' + \boldsymbol{\kappa'}_\perp(\varphi_\circ), \tag{84}$$

where $\pi(\varphi')$ is an arbitrary nonzero continuous scalar function (we remember that $\frac{\mathrm{d}\kappa'_\perp}{\mathrm{d}\varphi}$ cannot vanish). Finally, we substitute (84) and (82) into (78) to obtain

$$\left(\sinh\xi\sqrt{\kappa'^2_\perp + \frac{3}{2}\frac{w_\circ}{\tilde{\rho}_\circ^2}\tilde{\rho}} \mp \cosh\xi\sqrt{\frac{w_\circ}{2\tilde{\rho}_\circ^2}\tilde{\rho}}\right) = -\pi(\varphi). \tag{85}$$

Equations (83) and (85) are used to express both $\tilde{\rho}$ and ξ as functions of φ and this means that the problem is formally solved. Substitution of all the physical quantities obtained above into the Lorentz transformation (54) will provide all the quantities in the laboratory frame.

5. Symmetry analysis for the vortex mode equation

Symmetry investigations of physical equations usually provide useful tools to better understanding the behaviour of solutions and thus it is worth to apply this to our modal equations obtained in the previous sections. The sound mode appears to be very complicated and difficult to analyse and the entropy mode is relatively similar to the vortex mode. Therefore the only suitable typical equation for a detailed analysis is the vortex mode equation. The method presented in this section is based on Lie's method of infinitesimals that is a special case of the general method known as Cartan's equivalence problem. The equivalence problem even for the vortex mode equation (which is relatively a simple equation) appears to be very complex with too many variables and a wide variety of possible cases. So let us restrict our investigations to Lie symmetry analysis for the vortex mode equation.

Before starting this section, let us mention that from here on we change all the previous notations to quite new applications. So, we forget the meaning of all letters or symbols used in all the preceding sections and introduce new applications of them.

We consider equation (28) as a first-order ODE and rewrite it in the following form:

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t}\cdot\left(\mathbf{n} - \frac{\mathbf{k}\cdot\mathbf{n}}{\mathbf{k}^2 + w}\mathbf{k}\right) = 0, \tag{86}$$

where w is a constant, t is treated as the wave phase, and $\mathbf{k} = (k_1, k_2, k_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$ are some vectors in $\mathbb{R}^3$ having the physical meaning of $\boldsymbol{\kappa}$ and unit normal vector to the wavefront, respectively. We deal with the latter equation to find its point and contact symmetry properties and also give its fundamental invariants and a form of general solutions.

It is notable here that in the mathematical structure of the simple wave solution, the functional form of the unit normal vector $\mathbf{n}$ (see equation (13)) cannot be determined from the obtained equations. Thus, $\mathbf{n}$ must be arbitrarily fixed with the only restriction that its length be unit. This arbitrariness of $\mathbf{n}$ provides a wide freedom for us to choose this vector. Therefore, there appears two 'viewpoints' for the symmetry analysis of equation (86). The first is that we fix the form of $\mathbf{n}$ in equation (86) from the beginning and consider $\mathbf{k}$ as the only dependent variables. This type of analysis obviously depends on the special selected form of $\mathbf{n}$ and is not of our interest here (in section 3.1 a typical solution was given by this approach).

The 'second viewpoint' is to consider both $\mathbf{k}$ and $\mathbf{n}$ as dependent variables in equation (86) and find special symmetries consistent with this viewpoint. In other words, we may use the arbitrariness of $\mathbf{n}$ and impose some restriction on it to become consistent with the second viewpoint through which $\mathbf{n}$ will have some relation with $\mathbf{k}$. This of course is not important since there is no preassumption about $\mathbf{n}$ here.

Equation (86) is homogeneous and linear with respect to $\mathbf{n}$, so this condition is not essential in obtaining any solution. This condition appears important only for the compatibility of the simple wave structure. Regarding this fact, we make our symmetry analysis in both cases of arbitrary length and unit length for $\mathbf{n}$ and compare the results with each other.

Throughout this section we assume that indices i, j vary between 1 and 3. Also each index of a function implies the derivation of the function with respect to it, unless specifically stated otherwise.

5.1. The point symmetry of the equation

To find symmetry group of equation (86) by the Lie infinitesimal method, we follow the method presented in [35]. Roughly speaking, we find infinitesimal generators as well as the Lie algebra structure of the symmetry group of that equation. In this subsection, we are concerned with the action of the point transformation group.

In fact, equation (86) is an algebraic relation among the variables of the 1-jet space, $J^1(\mathbb{R}, \mathbb{R}^6)$, with (local) coordinates $(t, \mathbf{k}, \mathbf{n}, \mathbf{q}, \mathbf{p})$ where $\mathbf{q} = \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t}$ and $\mathbf{p} = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t}$. The coordinates involve an independent variable t and six dependent variables k_i, n_j and their first derivatives q_i, p_j with respect to t, respectively $(1 \leqslant i, j \leqslant 3)$.

Let $\mathcal{M}$ be the total space of independent and dependent variables. The solution space of equation (86), (if it exists) is a sub-variety $S_\Delta \subset J^1(\mathbb{R}, \mathbb{R}^6)$ of the first-order jet bundle of one-dimensional sub-manifolds of $\mathcal{M}$, that is, graph of functions k_i, n_j, of elements $(t, k_i(t), n_j(t))$, satisfying equation (86) and the relations $q_i = \frac{\partial k_i}{\partial t}$ and $p_j = \frac{\partial n_j}{\partial t}$ are all fulfilled.

We define a point transformation on $\mathcal{M}$ with relations

$$\tilde{t} = \phi(t, \mathbf{k}, \mathbf{n}), \qquad \tilde{k}_r = \chi_r(t, \mathbf{k}, \mathbf{n}), \qquad \tilde{n}_s = \psi_s(t, \mathbf{k}, \mathbf{n}),$$

where ϕ, χ_r and ψ_s are arbitrary smooth functions and $1 \leqslant r, s \leqslant 3$.

Theorem 1. *The set of all point infinitesimal generators in the form*

$$v_T := T\frac{\partial}{\partial t} + \sum_{i=1}^{3}\left\{\left(\mathbf{k}^2 + w - k_i^2\right)^{-1}\left(n_i - \frac{\mathbf{k}\cdot\mathbf{n}}{\mathbf{k}^2 + w}k_i\right)T_t\right\}\frac{\partial}{\partial n_i}, \tag{87}$$

is an infinite-dimensional Lie algebra of equation (86) for arbitrary **n** *(not necessarily unit).*

Proof. Let

$$v := T\frac{\partial}{\partial t} + \sum_{i=1}^{3}\left(K_i\frac{\partial}{\partial k_i} + N_i\frac{\partial}{\partial n_i}\right) \tag{88}$$

be the general form of infinitesimal generators that signify the Lie algebra $\mathfrak{g}$ of the symmetry group G of equation (86). In this relation, T, K_i and N_j are smooth functions of variables t, k_i and n_j. The first-order prolongation [35] of v is as follows:

$$v^{(1)} := v + \sum_i K_i^t\frac{\partial}{\partial q_i} + \sum_j N_j^t\frac{\partial}{\partial p_j},$$

where $K_i^t = D_t\, Q_1^i + T\, q_{i,t}$ and $N_j^t = D_t\, Q_2^j + T\, p_{j,t}$, in which D_t is the total derivative and $Q_1^i = K_i - T\, q_i$ and $Q_2^j = N_j - T\, p_j$ are characteristics of the vector field v [35]. By applying $v^{(1)}$ on (86), we obtain the following relation:

$$\sum_i\left\{\left[K_i((\mathbf{k}^2 + w)(\mathbf{k}\cdot\mathbf{n} + n_i) - 2\,k_i) + N_i(\mathbf{k}^2 + w)\left(\mathbf{k}^2 + w - k_i^2\right)\right]q_i + (n_i(\mathbf{k}^2 + w)\right.$$
$$\left.-k_i(\mathbf{k}\cdot\mathbf{n}))\left[K_{i\,t} - \sum_j\left(q_j\,K_{i\,k_j} + p_j\,K_{i\,n_j}\right) - q_i\left(T_t - \sum_j\left(q_j\,T_{k_j} + p_j\,T_{n_j}\right)\right)\right]\right\} = 0, \tag{89}$$

whenever equation (86) is satisfied. We may prescribe t, k_i, n_j, q_r, p_s $(1 \leqslant i, j, r, s \leqslant 3)$ arbitrarily while functions T, K_i and N_j only depend on t, k_i, n_j. Thus, equation (89) will be satisfied if and only if we have the following equations:

$$\sum_i K_{i\,t}\,(n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n})) = 0, \tag{90}$$

$$N_i(\mathbf{k}^2 + w)\big(\mathbf{k}^2 + w - k_i^2\big) - \sum_j K_{j\,k_i}\,(n_j(\mathbf{k}^2 + w) - k_j(\mathbf{k}\cdot\mathbf{n}))$$
$$- T_t\,(n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n})) + K_i((\mathbf{k}^2 + w)(\mathbf{k}\cdot\mathbf{n} + n_i) - 2\,k_i) = 0, \tag{91}$$

$$T_{k_i}(n_j(\mathbf{k}^2 + w) - k_j(\mathbf{k}\cdot\mathbf{n})) = 0, \tag{92}$$

$$T_{n_i}(n_j(\mathbf{k}^2 + w) - k_j(\mathbf{k}\cdot\mathbf{n})) = 0, \tag{93}$$

$$\sum_j K_{j\,n_i}\,(n_j(\mathbf{k}^2 + w) - k_j(\mathbf{k}\cdot\mathbf{n})) = 0. \tag{94}$$

These equations are called the determining equations. From equation (91) for each i we have

$$N_i = \big(\mathbf{k}^2 + w - k_i^2\big)^{-1}\Bigg\{(2\,k_i(\mathbf{k}^2 + w)^{-1} - (\mathbf{k}\cdot\mathbf{n} + n_i))\,K_i$$
$$+ \sum_j (n_j - k_j(\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w)^{-1})\,K_{j\,k_i} + (n_i - k_i(\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w)^{-1})\,T_t\Bigg\}. \tag{95}$$

Since $\mathbf{n} \neq 0$, without loss of generality, one may assume that $n_1 \neq 0$. Also, since $\mathbf{k}^2 + w \neq 0$, so by equations (92) and (93) we conclude that T just depends on t:

$$T = T(t). \tag{96}$$

By solving equation (90) with respect to t, we deduce the following relation of K_i's:

$$\sum_i (n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n}))\,K_i = 0. \tag{97}$$

After differentiating the latter equation with respect to n_j when we apply equations (94) we arrive at the following relations:

$$\big(\mathbf{k}^2 + w - k_i^2\big)K_i - \sum_{j\neq i} k_i k_j K_j = 0.$$

These relations suggest the general forms of K_1, K_2 and K_3 as follows:

$$K_1 = K_2 = K_3 = 0. \tag{98}$$

By applying (98) on relations (95) for different values of i, the forms of N_i's are also achieved:

$$N_i = \big(\mathbf{k}^2 + w - k_i^2\big)^{-1}\left(n_i - \frac{\mathbf{k}\cdot\mathbf{n}}{\mathbf{k}^2 + w}\,k_i\right)T_t. \tag{99}$$

Finally, the general form of infinitesimal generators as elements of point symmetry algebra of equation (86), which we call *point infinitesimal generators*, for arbitrary functions T is as introduced in relation (87). □

The Lie bracket (commutator) of every two vector fields in the form of (87) straightforwardly is an infinitesimal operator in the same form of them. More explicitly, the commutator of operators v_T and $v_{\overline{T}}$ is the vector field $v_{T\,\overline{T}_t - T_t\,\overline{T}}$. Hence, the Lie algebra

$\mathfrak{g} = \langle v_T \rangle$ of the point symmetry group G, when T is an arbitrary smooth function which depends on t, is a Lie algebra.

Theorem 2. *A complete set of functionally independent invariants of the one-parameter group generated by vector field (87) consists of*

$$I_i(t, \mathbf{k}, \mathbf{n}) = k_i, \qquad I_{i+3}(t, \mathbf{k}, \mathbf{n}) = |T|^{-\Lambda_i} \left(\mathbf{L}^{(i)} \cdot \mathbf{n}\right), \tag{100}$$

where $\mathbf{L}^{(i)}$ *s* ($i = 1, 2, 3$) *are functions of* $\mathbf{k}$ *to be determined below and*

$$\begin{aligned}
A &= \sum_{j=1}^{3} \left(\mathbf{k}^2 + w - k_j^2\right)k_j^2, & \Lambda_1 &= (\mathbf{k}^2 + w)^{-1}\left[1 + \sqrt{\frac{A}{3B}}\cos\left(\frac{\phi}{3} - \frac{\pi}{3}\right)\right], \\
B &= \prod_{j=1}^{3} \left(\mathbf{k}^2 + w - k_j^2\right)k_j^2, & \Lambda_2 &= (\mathbf{k}^2 + w)^{-1}\left[1 - 2\sqrt{\frac{A}{3B}}\cos\left(\frac{\phi}{3}\right)\right], \\
\phi &= \arctan\sqrt{\frac{A^3}{27\,B} - 1}, & \Lambda_3 &= (\mathbf{k}^2 + w)^{-1}\left[1 + 2\sqrt{\frac{A}{3B}}\cos\left(\frac{\phi}{3} + \frac{\pi}{3}\right)\right].
\end{aligned} \tag{101}$$

Proof. According to theorem 2.74 of [35], the invariants $u = I(t, \mathbf{k}, \mathbf{n})$ of one-parameter group with infinitesimal generators in the form of (87) satisfy the linear homogeneous partial differential equations of first order:

$$v[I] = 0.$$

The solutions of the latter are found by the method of characteristics (see [35, 36] for details). So, we can replace the above equation by the following characteristic system of ordinary differential equations:

$$\frac{\mathrm{d}t}{T} = \frac{\mathrm{d}k_i}{K_i} = \frac{\mathrm{d}n_j}{N_j}. \tag{102}$$

Replacing the coefficients from equation (87), one can conclude that k_i's are invariant, that is, if c_i are arbitrary constants, then one has the following invariants for $i = 1, 2, 3$:

$$I_i(t, \mathbf{k}, \mathbf{n}) = k_i = c_i. \tag{103}$$

From the characteristic equations one can conclude that considering the following system of the differential generator (87):

$$\frac{\mathrm{d}t}{T} = \frac{\mathrm{d}n_i}{N_i}, \qquad \text{for} \quad i = 1, 2, 3, \tag{104}$$

one can rewrite it as

$$(\mathbf{k}^2 + w)\,\frac{\mathrm{d}n_i}{\mathrm{d}\ln|T|} = n_i - \frac{\mathbf{k}\cdot\mathbf{n} - k_i\,n_i}{\mathbf{k}^2 + w - k_i^2}\,k_i, \qquad \text{for} \quad i = 1, 2, 3, \tag{105}$$

or as the following summary form:

$$\frac{\mathrm{d}\,\mathbf{n}}{\mathrm{d}\ln|T|} = M \cdot \mathbf{n} = \frac{1}{\mathbf{k}^2 + w}\begin{pmatrix} 1 & \frac{-k_1 k_2}{\mathbf{k}^2+w-k_1^2} & \frac{-k_1 k_3}{\mathbf{k}^2+w-k_1^2} \\ \frac{-k_1 k_2}{\mathbf{k}^2+w-k_2^2} & 1 & \frac{-k_2 k_3}{\mathbf{k}^2+w-k_2^2} \\ \frac{-k_1 k_3}{\mathbf{k}^2+w-k_3^2} & \frac{-k_2 k_3}{\mathbf{k}^2+w-k_3^2} & 1 \end{pmatrix} \cdot \mathbf{n}. \tag{106}$$

The latter equation can be solved for $\mathbf{n}$ by finding the eigenvalues of the matrix M (this result is true since the operator $\frac{\mathrm{d}}{\mathrm{d}\ln|T|}$ is linear). Solving the characteristic equation $\det(\Lambda\,\mathrm{Id}_3 - M) = 0$ which has three real roots, we find the eigenvalues Λ_i as

$$\begin{aligned}
\Lambda_1 &= (\mathbf{k}^2 + w)^{-1}\left[1 + C_1^{1/3} + C_2^{1/3}\right], \\
\Lambda_2 &= (\mathbf{k}^2 + w)^{-1}\left[1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}\right)C_1^{1/3} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i}\right)C_2^{1/3}\right],
\end{aligned} \tag{107}$$

$$\Lambda_3 = (\mathbf{k}^2 + w)^{-1} \left[1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i} \right) C_1^{1/3} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} \right) C_2^{1/3} \right],$$

where we have

$$\begin{aligned} A &= \sum_{j=1}^{3} \left(\mathbf{k}^2 + w - k_j^2\right)k_j^2, \qquad B = \prod_{j=1}^{3} \left(\mathbf{k}^2 + w - k_j^2\right)k_j^2, \\ C_1 &= -\frac{1}{B}[1 - \sqrt{1 - A^3/27B}], \qquad C_2 = -\frac{1}{B}[1 + \sqrt{1 - A^3/27B}]. \end{aligned} \tag{108}$$

Since each term $\left(\mathbf{k}^2 + w - k_j^2\right)k_j^2$ is non-negative, from a well-known inequality concerning arithmetic and geometric mean values, one finds that $\frac{A^3}{B} \geqslant 27$ and so C_1 is derived to be

$$C_1 = -\frac{1}{B}\left[1 - \mathrm{i}\sqrt{\frac{A^3}{27B} - 1}\right] = \left(\frac{A}{3B}\right)^{3/2} \mathrm{e}^{(\pi - \phi)\,\mathrm{i}}, \tag{109}$$

where $\phi = \arctan\sqrt{\frac{A^3}{27B} - 1}$. A similar method for C_2 tends to the following relation for the same value of ϕ:

$$C_2 = -\frac{1}{B}\left[1 - \mathrm{i}\sqrt{\frac{A^3}{27B} - 1}\right] = \left(\frac{A}{3B}\right)^{3/2} \mathrm{e}^{(\phi - \pi)\,\mathrm{i}}. \tag{110}$$

Thus

$$\Lambda_1 = (\mathbf{k}^2 + w)^{-1} \left[1 + \sqrt{\frac{A}{3B}} \cos\left(\frac{\phi}{3} - \frac{\pi}{3}\right) \right]. \tag{111}$$

Also using the last forms of C_1, C_2 for Λ_2 and Λ_3 in (107) we conclude that

$$\begin{aligned} \Lambda_2 &= (\mathbf{k}^2 + w)^{-1} \left[1 - 2\sqrt{\frac{A}{3B}} \cos\left(\frac{\phi}{3}\right) \right], \\ \Lambda_3 &= (\mathbf{k}^2 + w)^{-1} \left[1 + 2\sqrt{\frac{A}{3B}} \cos\left(\frac{\phi}{3} + \frac{\pi}{3}\right) \right]. \end{aligned} \tag{112}$$

Now equation (106) can be considered as

$$\frac{\mathrm{d}}{\mathrm{d}\ln|T|}\,(\mathbf{L}^{(i)} \cdot \mathbf{n}) = \Lambda_i\,(\mathbf{L}^{(i)} \cdot \mathbf{n}), \qquad i = 1, 2, 3, \tag{113}$$

where $\mathbf{L}^{(i)}$ is the left eigenvector of the matrix M corresponding to the eigenvalue Λ_i. Equations (113) are justified since $\mathbf{L}^{(i)}$'s are only functions of $\mathbf{k}$ as an invariant. The forms of $\mathbf{L}^{(i)}$'s in terms of $\mathbf{k}$ are too complicated and thus we do not write their explicit expressions here. The above equations result in relations $\mathbf{L}^{(i)} \cdot \mathbf{n} = d_i\,|T|^{\Lambda_i}$ where d_i's are arbitrary constants ($i = 1, 2, 3$). Thus we deduce the following invariants:

$$I_{i+3}(t, \mathbf{k}, \mathbf{n}) := |T|^{-\Lambda_i}\,(\mathbf{L}^{(i)} \cdot \mathbf{n}) = d_i, \qquad i = 1, 2, 3. \tag{114}$$

Finally from [35], p 62, we find that the functions $I_1, I_2, \ldots, I_6$ form a complete set of functionally independent invariants of the one-parameter group of the vector field (87). □

Similar to the theorem of section 4.3.3 of [36], the derived invariants (100) as independent first integrals of the characteristic system of the infinitesimal generator (87) provide the general solution

$$S(t, \mathbf{k}, \mathbf{n}) := \mu(I_1(t, \mathbf{k}, \mathbf{n}), I_2(t, \mathbf{k}, \mathbf{n}), \ldots, I_6(t, \mathbf{k}, \mathbf{n})),$$

with an arbitrary function μ, which satisfies in the equation $v[\mu] = 0$. This theorem can be extended for each finite set of independent first integrals (invariants) of characteristic system provided with an infinitesimal generator.

In the following, we give some examples provided with different selections of coefficients of equation (87) to show the method explicitly. We always assume the existence of nonzero coefficients of the vector fields.

Example 1. If we assume that $T = c = \text{const}$, then the infinitesimal operator (87) reduces to the vector field $v_1 = c\,\frac{\partial}{\partial t}$ and the group transformations (or flows) for the parameter s are expressible as $(t, \mathbf{k}, \mathbf{n}) \to (\tilde{t}^{(s)}, \tilde{\mathbf{k}}^{(s)}, \tilde{\mathbf{n}}^{(s)}) = (t + cs, \mathbf{k}, \mathbf{n})$ which form the (local) symmetry group of v_1. The derived invariants in this case will be as follows:

$$I_i = k_i, \qquad I_{i+3} = n_i, \qquad \text{for } i = 1, 2, 3.$$

Therefore, the general solution corresponding to v_1 when μ is an arbitrary function, is

$$S(t, \mathbf{k}, \mathbf{n}) = \mu(\mathbf{k}, \mathbf{n}).$$

Example 2. Let $T = t$, then the infinitesimal generator is

$$v_2 = t\frac{\partial}{\partial t} + \sum_{j=1}^{3} \left(\mathbf{k}^2 + w - k_j^2\right)^{-1} \left(n_j - \frac{\mathbf{k}\cdot\mathbf{n}}{\mathbf{k}^2 + w} k_j\right) \frac{\partial}{\partial n_j}.$$

Thus, the flows of v_2 for various values of parameter s are

$$(t, k_i, n_j) \longrightarrow \left(\tilde{t}^{(s)}, \tilde{k}_i^{(s)}, \tilde{n}_j^{(s)}\right)$$
$$= \left(t\mathrm{e}^{s}, k_i, \left(\mathbf{k}^2 + w - k_j^2\right)^{-1}\left[n_j \mathrm{e}^{\frac{s}{\mathbf{k}^2+w}} + (\mathbf{k}\cdot\mathbf{n} - k_j n_j)k_j\left(1 - \mathrm{e}^{\frac{s}{\mathbf{k}^2+w}}\right)\right]\right).$$

Also, we have the invariants

$$I_i = k_i, \qquad I_{i+3} = |t|^{-\Lambda_i}(\mathbf{L}^{(i)}\cdot\mathbf{n}), \qquad \text{for} \quad i = 1, 2, 3,$$

whenever defined, and the general solution of equation (86) as

$$S(t, \mathbf{k}, \mathbf{n}) = \mu(\mathbf{k}, |t|^{-\Lambda_i}(\mathbf{L}^{(i)}\cdot\mathbf{n})),$$

where μ is an arbitrary function.

Example 3. In the case $T = \mathrm{e}^t$ the infinitesimal generator (87) changes to

$$v_3 = \mathrm{e}^t\left[\frac{\partial}{\partial t} + \sum_{j=1}^{3}\left(\mathbf{k}^2 + w - k_j^2\right)^{-1}\left(n_j - \frac{\mathbf{k}\cdot\mathbf{n}}{\mathbf{k}^2 + w}k_j\right)\frac{\partial}{\partial n_j}\right],$$

with group transformations of the parameter s transforming (t, k_i, n_j) to

$$\left(\tilde{t}^{(s)}, \tilde{k}_i^{(s)}, \tilde{n}_j^{(s)}\right)$$
$$= \left(\ln\{\mathrm{e}^t(1 - s\mathrm{e}^t)^{-1}\}, k_i, \left(\mathbf{k}^2 + w - k_j^2\right)^{-1}\left[n_j \mathrm{e}^{\frac{\mathrm{e}^t s}{\mathbf{k}^2+w}} + (\mathbf{k}\cdot\mathbf{n} - k_j n_j)k_j\left(\mathrm{e}^{-t} - \mathrm{e}^{\frac{\mathrm{e}^t s}{\mathbf{k}^2+w}}\right)\right]\right),$$

wherever defined. Independent invariants are

$$I_i = k_i, \qquad I_{i+3} = \mathrm{e}^{-\Lambda_i t}(\mathbf{L}^{(i)}\cdot\mathbf{n}), \qquad \text{for} \quad i = 1, 2, 3,$$

and hence the general solution of (86) with respect to infinitesimal operator v_3 is an arbitrary function of these invariants.

In the above examples we saw that if $(t, \mathbf{k}, \mathbf{n})$ is a solution of equation (86) then also for any arbitrary value of s, $(\tilde{t}^{(s)}, \tilde{\mathbf{k}}^{(s)}, \tilde{\mathbf{n}}^{(s)})$ is another solution for this equation. Any solution presents **k** and **n** as functions of the wave phase t. Therefore, the above symmetry suggests that

$$\frac{\mathrm{d}\tilde{\mathbf{k}}^{(s)}}{\mathrm{d}\tilde{t}^{(s)}} \cdot \left(\tilde{\mathbf{n}}^{(s)} - \frac{\tilde{\mathbf{k}}^{(s)} \cdot \tilde{\mathbf{n}}^{(s)}}{\tilde{\mathbf{k}}^{(s)\,2} + w}\,\tilde{\mathbf{k}}^{(s)}\right) = 0. \qquad (115)$$

Physically this means as follows: assume a special form of $\mathbf{n}(t)$ is given as a function of the wave phase t which we have successfully found a solution $\mathbf{k}(t)$ related to it based on the relevant symmetry restrictions. Equation (115) implies that there is a solution, namely, $\tilde{\mathbf{k}}^{(s)}$ for the new form of $\mathbf{n}$, namely, $\tilde{\mathbf{n}}^{(s)}$. In other words, any solution of the vortex mode equation obtained through the above symmetry discussions, provides a solution for another equation corresponding to it.

Up to now, we have performed our analysis according to the 'second viewpoint' introduced in the paragraph after equation (86), through which **k** and **n** are both considered as dependent variables so that a relation between them may appear via special obtained solutions. In such a viewpoint we have dropped the restriction of the unit length for **n** which we have seen is not important because equation (86) is linear and homogeneous with respect to **n**. However, let us add this restriction to the above symmetry analysis reaching the following theorem and its corollary.

Theorem 3. *The point Lie algebra of equation (86) when* **n** *is a unit (constant) normal vector to the wavefront is* $\mathfrak{g} = \langle \frac{\partial}{\partial t} \rangle$ *isomorphic to the Lie algebra* $\mathbb{R}$. *Therefore, the point symmetry group of the equation with this additional condition is the group of phase translations.*

Proof. When we assume **n** to be of unit length, then by the action of $v^{(1)}$ (the first prolongation of the general form (88) of the infinitesimal generator v) on the relation $n_1^2 + n_2^2 + n_3^2 = 1$ we tend to the following equation:

$$n_1 N_1 + n_2 N_2 + n_3 N_3 = 0, \qquad (116)$$

which must be added to the previous derived conditions (96), (98) and (99). The last equation along with the deduced form of N_i's in (99) implies that $T_t = 0$. Hence, $T = c$ for arbitrary constant c and for each i, $N_i = 0$. Therefore, the form of infinitesimal generators reduces from relation (87) to $v = \frac{\partial}{\partial t}$. □

According to [35], p 209, any system of partial differential equations which has only a finite-dimensional symmetry group is certainly not linearizable, that is, for every change of variables, it cannot be mapped to an inhomogeneous form of the linear system $\mathcal{D}[u] = f$, where $\mathcal{D}$ is a second-order linear differential operator, u indicates dependent variables and f denotes smooth functions of independent variables.

Corollary 1. *The vortex mode equation in the form (86) with the constant length for* **n** *cannot be reduced into an inhomogeneous form of a linear equation.*

5.2. Invariant differential equations

In the last subsection, in examples 1–3, we suggested some different forms of general solutions under conditions of those examples. Let $V(t, k_i, n_j) = 0$, as an unknown function of seven variables in $J^0(\mathbb{R}, \mathbb{R}^6)$, be the implicit relation which defines k_i's and n_j's as functions of t. Then by the theorem of section 4.3.3 in [36], one may replace the equation $V = 0$ by the equation $\mu(I_1, \ldots, I_6) = 0$ where I_k's are functionally independent invariants of the vortex mode equation and μ is an arbitrary function. Thus, in examples 1–3, by considering various forms of μ which define explicit relations among the independent and dependent variables, one can extract special exact solutions of equation (86). Cases I–IV below are considered under the situation of example 1, and case V is related to examples 2 and 3.

Case I. If $\mu = (\mathbf{k} \pm \mathbf{n})^2 = 0$, then substituting the solution $\mathbf{k} = \pm\mathbf{n}$ in the equation, it reduces to the below expression:

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} \cdot \mathbf{n} = 0,$$

which shows that the length of $\mathbf{n}$ and consequently the length of $\mathbf{k}$ is constant (remember the discussion at the beginning of this section in which temporarily we ignored the unity of the length of $\mathbf{n}$). From the physical point of view, this solution of equation (86) is obtained when $\mathbf{k}$ is in the direction of $\mathbf{n}$ or in the opposite direction of it, which yields that streamlines are everywhere perpendicular to wavefronts.

Case II. Let $\mu = \mathbf{k} \cdot \mathbf{n} = 0$; then the principal equation reduces to the following form:

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} \cdot \mathbf{n} = 0.$$

It is worth to note the physical meaning of the condition $\mathbf{k} \cdot \mathbf{n} = 0$. According to equations (13) and (22), we see that in this special case, the phase velocity $\lambda = 0$. Thus, our simple wave solution reduces to a stationary solution. In addition, since $\mathbf{k}$ and $\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t}$ are both perpendicular to $\mathbf{n}$, we see that each streamline (which coincides the integral curve of $\mathbf{k}$) starting from a point at a wavefront, always remains in the same wavefront.

Case III. When $\mu = |\mathbf{k} \times \mathbf{n}| = 0$, then we find it as a generalization of case I, where $\mathbf{k}$ is still parallel to $\mathbf{n}$. Therefore, we see that streamlines are perpendicular to wavefronts.

Case IV. For different values $1 \leqslant \alpha, \beta, \gamma \leqslant 3$ assume $\mu = k_\alpha^2 + k_\beta^2 = 0$. Then the equation will change to the following equation $(1 \leqslant \gamma \leqslant 3)$:

$$\frac{n_\gamma w}{k_\gamma^2 + w} \frac{\mathrm{d}k_\gamma}{\mathrm{d}t} = 0,$$

with the exact solution $k_\gamma = \text{const}$. This condition imposes a relation between the three components of the fluid velocity $\mathbf{v}$.

Now, we consider the general form of examples 2 and 3 when T is an arbitrary function of t and μ provides an implicit solution of equation (86) in terms of $I_1, \ldots, I_6$ in (100). In this condition, one can consider the following case for μ.

Case V. Let $\mu = \sum_i |T|^{-\Lambda_i} (\mathbf{L}^{(i)} \cdot \mathbf{n}) = 0$. Then one concludes that

$$\left(\sum_i |T|^{-\Lambda_i} \mathbf{L}^{(i)}\right) \cdot \mathbf{n} = 0. \tag{117}$$

The expression in the parentheses is a vector function of $\mathbf{k}$ which is seen to be perpendicular to $\mathbf{n}$ at any point of the physical space. This vector is therefore tangent to the wavefront everywhere. Restriction (117) is a generalization of the restriction proposed in case II with more complexity.

Thus, we are faced with a very wide variety of arbitrary forms of μ each of which may impose a special restriction or condition on the physical solution. So, we observe the possibility of creating a lot of classical solutions of which we just selected five cases having some physical interests.

Table 1. Commutator table provided by contact symmetries.

	v_T	v_{K_α}	v_{N_β}	v_{Q_γ}	v_{P_η}
v_T	0	$v_T + v_{K_\alpha}$	$v_T + v_{N_\beta}$	$v_T + v_{Q_\gamma}$	$v_T + v_{P_\eta}$
v_{K_α}	$-v_T - v_{K_\alpha}$	0	$v_{K_\alpha} + v_{N_\beta}$	$v_{K_\alpha} + v_{Q_\beta}$	$v_{K_\alpha} + v_{P_\eta}$
v_{N_β}	$-v_T - v_{N_\beta}$	$-v_{K_\alpha} - v_{N_\beta}$	0	$v_{N_\beta} + v_{Q_\gamma}$	$v_{N_\beta} + v_{P_\eta}$
v_{Q_γ}	$-v_T - v_{Q_\gamma}$	$-v_{K_\alpha} - v_{Q_\gamma}$	$-v_{N_\beta} - v_{Q_\gamma}$	0	$v_{Q_\gamma} + v_{P_\eta}$
v_{P_η}	$-v_T - v_{P_\eta}$	$-v_{K_\alpha} - v_{P_\eta}$	$-v_{N_\beta} - v_{Q_\eta}$	$-v_{Q_\gamma} - v_{P_\eta}$	0

5.3. The contact symmetry of the equation

In the last subsection, we did not take into account the role of derivatives in the presented symmetry analysis. It is natural to investigate the contact symmetry analysis including the derivatives as well. According to Bäcklund's theorem [35], if the number of dependent variables is greater than 1 (like our problem), then each contact transformation is the prolongation of a point transformation. In this subsection, we present the structure of infinitesimal generators of contact transformations and their related flows. The normal vector to wavefront, **n**, is assumed to be either arbitrary (not necessarily unit) or of unit length and then we find the contact symmetry properties of equation (86) in these two situations.

We suppose that the general form of a contact transformation is of the form

$$\widetilde{t} = \phi(t, k_i, n_j, q_r, p_s), \qquad \widetilde{k_l} = \chi_l(t, k_i, n_j, q_r, p_s), \qquad \widetilde{n}_m = \psi_m(t, k_i, n_j, q_r, p_s),$$
$$\widetilde{q}_n = \eta_n(t, k_i, n_j, q_r, p_s), \qquad \widetilde{p}_u = \zeta_u(t, k_i, n_j, q_r, p_s),$$

where i, j, l, m, n and u varies between 1 and 6, and ϕ, χ_l, ψ_m, η_n and ζ_u are arbitrary smooth functions.

Theorem 4. *The contact symmetry group of equation (86) is an infinite-dimensional Lie algebra generated by the contact infinitesimal operators*

$$v_T = T\,\frac{\partial}{\partial t}, \qquad v_{K_i} = K_i\left(\frac{\partial}{\partial k_i} + \frac{n_i(\mathbf{k}^2+w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2+w-2\,k_i)}{(\mathbf{k}^2+w)(n_1(\mathbf{k}^2+w) - k_1(\mathbf{k}\cdot\mathbf{n}))}\,q_i\,\frac{\partial}{\partial q_1}\right),$$
$$v_{P_i} = P_i\,\frac{\partial}{\partial p_i}, \qquad v_{N_i} = N_i\left(\frac{\partial}{\partial n_i} - \frac{\mathbf{k}^2+w-k_i^2}{n_1(\mathbf{k}^2+w) - k_1(\mathbf{k}\cdot\mathbf{n})}\,q_i\,\frac{\partial}{\partial q_1}\right), \tag{118}$$
$$v_{Q_j} = Q_j\left(\frac{\partial}{\partial q_j} - \frac{n_j(\mathbf{k}^2+w) + k_j(\mathbf{k}\cdot\mathbf{n})}{n_1(\mathbf{k}^2+w) - k_1(\mathbf{k}\cdot\mathbf{n})}\,\frac{\partial}{\partial q_1}\right),$$

where for $1 \leqslant i \leqslant 3$ and $2 \leqslant j \leqslant 3$, T, K_i, N_j, Q_j, P_i are arbitrary smooth functions. Also, the Lie bracket of two such vector fields is in the form introduced in the commutator table 1.

Proof. In this case of group action an infinitesimal generator, which is a vector field in $J^1(\mathbb{R}, \mathbb{R}^6)$, has the following general form:

$$v := T\,\frac{\partial}{\partial t} + \sum_{i=1}^{3}\left\{K_i\frac{\partial}{\partial k_i} + N_i\frac{\partial}{\partial n_i} + Q_i\frac{\partial}{\partial q_i} + P_i\frac{\partial}{\partial p_i}\right\}, \tag{119}$$

for arbitrary smooth functions T, K_l, N_m, Q_m, P_u ($l = 1, 2$ and $1 \leqslant m, n, u \leqslant 3$).

Since our computations are done in 1-jet space, we do not need to lift v to higher jet spaces and hence we act v (itself) on equation (86). Accordingly, we obtain the relation

$$\sum_{i=1}^{3}\left\{N_i q_i(\mathbf{k}^2+w)\big(\mathbf{k}^2+w-k_i^2\big) + Q_i(\mathbf{k}^2+w)[n_i(\mathbf{k}^2+w) - k_i(\mathbf{k}\cdot\mathbf{n})]\right.$$
$$\left. - K_i q_i\big[n_i(\mathbf{k}^2+w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2+w-2k_i)\big]\right\} = 0.$$

From $\mathbf{n} \neq 0$, without loss of generality, we can suppose that $n_1 \neq 0$; then the solution to this equation would be

$$Q_1 = (\mathbf{k}^2 + w)^{-1}(n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n}))^{-1}\left\{\sum_{i=1}^{3}\left[K_i q_i (n_i(\mathbf{k}^2 + w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w - 2k_i)) - N_i q_i (\mathbf{k}^2 + w)\left(\mathbf{k}^2 + w - k_i^2\right)\right] - \sum_{j=2}^{3} Q_i (\mathbf{k}^2 + w)[n_i(\mathbf{k}^2 + w) - k_i(\mathbf{k}\cdot\mathbf{n})]\right\}.$$

Therefore, the infinitesimal generator, which we refer to as a *contact infinitesimal generator*, is of the form

$$v = T\frac{\partial}{\partial t} + \sum_i K_i\left(\frac{\partial}{\partial k_i} + \frac{n_i(\mathbf{k}^2 + w) + (\mathbf{k}\cdot\mathbf{n})(\mathbf{k}^2 + w - 2\,k_i)}{(\mathbf{k}^2 + w)(n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n}))}\, q_i \frac{\partial}{\partial q_1}\right) + \sum_i N_i\left(\frac{\partial}{\partial n_i} - \frac{\mathbf{k}^2 + w - k_i^2}{n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n})}\, q_i \frac{\partial}{\partial q_1}\right) + \sum_i P_i \frac{\partial}{\partial p_i} + \sum_{j=2,3} Q_j\left(\frac{\partial}{\partial q_j} - \frac{n_j(\mathbf{k}^2 + w) + k_j(\mathbf{k}\cdot\mathbf{n})}{n_1(\mathbf{k}^2 + w) - k_1(\mathbf{k}\cdot\mathbf{n})}\frac{\partial}{\partial q_1}\right). \qquad (120)$$

One may divide the latter form into the vector fields (118) to consist a basis for the Lie algebra $\mathfrak{g} = \langle v\rangle$ of the contact symmetry group G. The commutator of every two vector fields (118) is a linear combination of two operators (118), which are generally in the form of those two operators again. Thus, these vector fields construct a basis for the Lie algebra $\mathfrak{g}$ of the contact symmetry group G. The commutator table is given in table 1 for $1 \leqslant \alpha, \beta, \eta \leqslant 3$ and $2 \leqslant \gamma \leqslant 3$. In this table, when the commutator of two vector fields is generally in the same form of some vector fields in (118), then we use those general forms again to show the results of commutators. □

It is interesting to note that in equations (118) if all the coefficients, T, K_i, N_i, Q_j, P_i, are equal to 1 and also $\mathbf{k}\cdot\mathbf{n} = 0$, then in this special case we obtain the following theorem for $\mathbf{n}$. The restriction $\mathbf{k}\cdot\mathbf{n} = 0$ has a relatively acceptable physical justification (refer to case II of section 5.2).

Theorem 5. *If* $(t, \mathbf{k}, \mathbf{n}, \mathbf{q}, \mathbf{p})$ *is a solution to the vortex mode equation (86) for* $\mathbf{k}\cdot\mathbf{n} = 0$, *then the equation holds for*

$$\begin{array}{ll} [G_T] & (t+s,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}),\\ [G_{K_i}] & (t,\ \mathbf{k}+S,\ \mathbf{n},\ q_1\mathrm{e}^{-n1s_1} - n_2q_2s_2 - n_3q_3s_3,\ q_2,\ q_3,\ \mathbf{p}),\\ [G_{N_1}] & \left(t,\ \mathbf{k},\ n_1+s,\ n_2,\ n_3,\ q_1 n_1^{A_1}\mathrm{e}^{-A_1\ln(s+n_1)},\ q_2,\ q_3,\ \mathbf{p}\right),\\ [G_{N_2}, G_{N_3}] & (t,\ \mathbf{k}+\bar{S},\ \mathbf{n},\ q_1 - A_2q_2s_2 - A_3q_3s_3,\ q_2,\ q_3,\ \mathbf{p}),\\ [G_{Q_j}] & \left(t,\ \mathbf{k},\ \mathbf{n},\ q_1 - \frac{n2}{n1}s_2 - \frac{n3}{n1}s_3,\ q_2+s_2,\ q_3+s_3,\ \mathbf{p}\right),\\ [G_{P_i}] & (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}+S), \end{array} \qquad (121)$$

where s, s_1, s_2, s_3 *are the arbitrary constant parameters,* $S = (s_1, s_2, s_3)$, $\bar{S} = (0, s_2, s_3)$ *and* $A_i = 1 - \frac{k_i^2}{\mathbf{k}^2 + w}$ *for* $i = 1, 2, 3$.

Proof. Let $G_T, G_{K_i}, G_{N_i}, G_{Q_j}, G_{P_i}$ resp. be one-parameter groups corresponding to contact infinitesimal operators $v_T, v_{K_i}, v_{N_i}, v_{Q_j}, v_{P_i}$ of (118); then we have

$$
\begin{aligned}
G_1 &:= G_T: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t+s,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}),\\
G_2 &:= G_{K_1}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ k_1+s,\ k_2,\ k_3,\ \mathbf{n},\ q_1\mathrm{e}^{-n_1 s},\ q_2,\ q_3,\ \mathbf{p}),\\
G_3 &:= G_{K_2}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ k_1,\ k_2+s,\ k_3,\ \mathbf{n},\ q_1,\ -n_2q_2s,\ q_3,\ \mathbf{p}),\\
G_4 &:= G_{K_3}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ k_1,\ k_2,\ k_3+s,\ \mathbf{n},\ q_1,\ q_2,\ -n_3q_3s,\ \mathbf{p}),\\
G_5 &:= G_{N_1}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ \mathbf{k},\ n_1+s,\ n_2,\ n_3,\ q_1 n_1^{A_1}\mathrm{e}^{-A_1\ln(s+n_1)},\ q_2,\ q_3,\ \mathbf{p}),\\
G_6 &:= G_{N_2}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ \mathbf{k},\ n_1,\ n_2+s,\ n_3,\ q_1 - A_2q_2s,\ q_2,\ q_3,\ \mathbf{p}),\\
G_7 &:= G_{N_3}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ \mathbf{k},\ n_1,\ n_2,\ n_3+s,\ q_1 - A_3q_3s,\ q_2,\ q_3,\ \mathbf{p}),\\
G_8 &:= G_{Q_2}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto \left(t,\ \mathbf{k},\ \mathbf{n},\ q_1 - \tfrac{n_2}{n_1}s,\ q_2+s,\ q_3,\ \mathbf{p}\right),\\
G_9 &:= G_{Q_3}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto \left(t,\ \mathbf{k},\ \mathbf{n},\ q_1 - \tfrac{n_3}{n_1}s,\ q_2,\ q_3+s,\ \mathbf{p}\right),\\
G_{10} &:= G_{P_1}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ p_1+s,\ p_2,\ p_3),\\
G_{11} &:= G_{P_2}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ p_1,\ p_2+s,\ p_3),\\
G_{12} &:= G_{P_3}: && (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ \mathbf{p}) \longmapsto (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{q},\ p_1,\ p_2,\ p_3+s),
\end{aligned}
\tag{122}
$$

for an arbitrary parameter s. Thus if $(t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{p})$ satisfies in the vortex mode equation (when $\mathbf{k}\cdot\mathbf{n} = 0$), then the actions $G_l \cdot (t,\ \mathbf{k},\ \mathbf{n},\ \mathbf{p})$ $(1 \leqslant l \leqslant 12)$ suggest new solutions for the equation and this completes the proof. □

Finally, according to the discussion just before theorem 3 (section 5.6) it looks useful to add the restriction of unit length for $\mathbf{n}$ to the present (contact) symmetry analysis.

Theorem 6. *The contact symmetry group of equation (86) consisting of unit normal wavefront is an infinite-dimensional Lie algebra and its Lie algebra is generated by the contact infinitesimal operators (120) when we replace the coefficient* N_1 *by* $-\frac{n_2}{n_1}N_2 - \frac{n_3}{n_1}N_3$. *The commutator table of these vector fields is in the form of table 1 when we eliminate the row and column corresponding to* v_{N_1} *and then change* v_{N_2} *and* v_{N_3} *to the new forms.*

Proof. One may repeat the above process for the problem of finding contact Lie algebra of equation (86) with the supplementary condition of $\mathbf{n}$ to be unit. In this case, there is another condition $v(equation(116)) = 2(n_1N_1 + n_2N_2 + n_3N_3) = 0$ by the action of a contact infinitesimal generator on equation (86) which must be added to other relations of coefficients. Since $\mathbf{n} \neq 0$, we can suppose $n_1 \neq 0$ and then $N_1 = -\frac{n_2}{n_1}N_2 - \frac{n_3}{n_1}N_3$. □

6. Summary and conclusions

Towards a deeper understanding of the mysterious behaviour of hydrodynamical equations, it is worthwhile to look for various classical solutions. Among these solutions, simple waves and multi-waves are very appropriate for compressible flows. These solutions show, in particular, how it is possible that smooth initial conditions convert to discontinuities and singularities at future times. Thus, there is a hope that by a detailed and deep analysis of these waves, one may find more general statements about the appearance of non-smoothness converging to weak solutions. As mentioned in the introduction, since 1D Riemann waves were useful to prove the 1D shock convergence, it looks natural to be able to generalize the case to multi-dimensional flows. In other words, one may use the present multi-dimensional classical solutions of simple waves to investigate the shock convergence in higher dimensions. To achieve this aim we should take into account viscous terms and utilize this multi-dimensional simple wave as a base to reach to a perhaps more general shock convergence. However, any classical solution definitely has its own value in understanding the behaviour of the system under consideration.

In the present work, a multi-dimensional version of simple waves introduced in [17, 19] were employed for fully relativistic fluids and plasmas. Each wavefront is a plane travelling with its own phase velocity vector. The intersection of different wavefronts is forbidden in the domain of the (classical) solution. Also, at each instant of time there is a surface as the boundary between the two regions: the region of the validity of the solution and the forbidden region where the solution does not exist. This boundary generally moves and changes in the course of time.

Similar to the non-relativistic case [19], three essential modes were found, namely the vortex, the entropy and sound modes. Each mode suggests a wide variety of classical solutions while only very few typical solutions were presented as illustrations of the method of solving the problem at hand. The vortex and entropy modes were solved both in the laboratory and wave frames while due to the high complexity of sound modes we studied them only in the wave frame. Furthermore, as a special physically valid example we considered the thermodynamically state equation at ultra-relativistic temperatures and obtained a formal classical solution in the wave frame.

A symmetry analysis for the vortex mode equation (as a typical equation) led to the characterization of point and contact infinitesimal generators as well as fundamental invariants of the equation. According to what has been mentioned in section 1, a complete symmetry analysis for fully relativistic fluid equations (6)–(9) is so difficult, wide-ranging and detailed, that it requires a separate investigation. Thus, here a limited symmetry analysis was pursued only for one of the modal equations that appeared in the framework of simple waves. The discussion of section 5.1 led us to obtain invariants of point transformations (which maps the space of solutions to itself) possessing two distinguished physical results. The first is that under certain restrictions we can extract a new equivalent equation from the original one with the same space of solutions. This means that given a suitable special solution, yields another solution for a new equation of the same type (see section 5.1, especially equation (115)). The second result concerns with the possibility of creating a wide class of solutions which only very few of them with physical interests were introduced via cases I–V in section 5.2.

As one step further than the point symmetry, the contact symmetry was investigated to find contact transformations mapping the space of solutions (as a submanifold of the first-order jet bundle related to the equation) to itself. The invariants corresponding to these transformations were too long and cumbersome to appear here. In the special case of our problem, the contact and point symmetry group of the vortex mode equation were both found to be infinite-dimensional Lie groups when the normal vector to the wavefront is not necessarily unit. It was discussed at the beginning of section 5 that the unit length of this normal vector is not important in the utility of our symmetry analysis. However, it was found that when it has the unit length, there appears a one-dimensional point symmetry group while the contact symmetry group is still infinite-dimensional. The same procedure can be probably made for equations of other modes, namely the entropy and sound modes.

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